

On the Computational Complexity of Defining Sets*

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Abstract

Suppose we have a family \mathcal{F} of sets. For every $S \in \mathcal{F}$, a set $D \subseteq S$ is a defining set for (\mathcal{F}, S) if S is the only element of \mathcal{F} that contains D as a subset. This concept has been studied in numerous cases, such as vertex colorings, perfect matchings, dominating sets, block designs, geodetics, orientations, and Latin squares.

In this paper, first, we propose the concept of a defining set of a logical formula, and we prove that the computational complexity of such a problem is Σ_2 -complete.

We also show that the computational complexity of the following problem about the defining set of vertex colorings of graphs is Σ_2 -complete:

INSTANCE: A graph G with a vertex coloring c and an integer k .

QUESTION: If $\mathcal{C}(G)$ be the set of all $\chi(G)$ -colorings of G , then does $(\mathcal{C}(G), c)$ have a defining set of size at most k ?

Moreover, we study the computational complexity of some other variants of this problem.

KEYWORDS: defining sets; complexity; graph coloring; satisfiability.

1 Introduction

In this paper we consider a unification of the concepts already known as critical sets, forcing sets, and defining sets, where we formulate different natural problems in this regard. Specially, through considering such problems for 3SAT, by introducing suitable reductions, we prove that the decision problem related to the minimum defining set problem of graph colorings¹ is Σ_2 -complete.

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¹Defined formally below

Defining sets were studied for Latin squares [3, 7], perfect matchings [1, 2, 11], orientations [5], geodetics [6], vertex colorings [13], designs [9], and dominating sets [4].

Let \mathcal{F} be a family of sets. For every $S \in \mathcal{F}$, a set $D \subseteq S$ is a **defining set** of (\mathcal{F}, S) , if S is the only element in \mathcal{F} which contains D as a subset. By abuse of language every defining set of (\mathcal{F}, S) is also called a defining set of \mathcal{F} .

In what follows, we try to introduce a general formulation for the type of problems we are going to consider in the rest of this paper.

Suppose an input I is given. The input I might be a graph, a number, or any other mathematical object. Then let $\mathcal{F}(I)$ be a family of sets which is defined according to the set I . In this paper we are interested in the computational complexity of the following three general types of questions for specified inputs and definitions of \mathcal{F} .

1. • Q1

INSTANCE: I , a set $S \in \mathcal{F}(I)$, and a set $D \subseteq S$.

QUESTION: Is D a defining set of $(\mathcal{F}(I), S)$?

2. • Q2

INSTANCE: I , a set $S \in \mathcal{F}(I)$, and an integer k .

QUESTION: Does S have a defining set of size at most k ?

3. • Q3

INSTANCE: I and an integer k .

QUESTION: Does $\mathcal{F}(I)$ have a defining set of size at most k ?

The computational complexity of the problems related to defining sets was first studied by Colbourn in [7]. He studied Q1 when $\mathcal{F}(n)$ is the set of Latin squares of order n , and proved that this question is **CoNP**-complete. Recently Adams, Mahdian, and Mahmoodian [1] studied Q2 when $\mathcal{F}(G)$ is the set of perfect matchings of a graph G , and proved that the question is **NP**-complete. In [2] it is shown that the question Q3 for this family is **NP**-complete. It is not hard to see that the question Q1 for this family is in **P**. Hatami and Tusserkani in [12] studied Q2 and Q3 when $\mathcal{F}(G)$ is the set of vertex colorings of a graph G , and proved that both of the questions are **NP**-hard. In this paper we improve their result by showing that these problems are both Σ_2 -complete. In this regard we consider the family of all proper assignments to the variables of a k CNF where a k CNF is a Boolean expression in conjunctive normal form such that every clause has exactly k variables. Let Q1- k SAT, Q2- k SAT, and Q3- k SAT stand for the three questions Q1, Q2, and Q3 in this case, respectively. We show that Q1-3SAT is **CoNP**-Complete, and Q2-3SAT and Q3-3SAT are both Σ_2 -complete. We also refer the reader to the recent paper [10] for some other computational complexity results on the defining sets of vertex colorings.

We determine the computational complexity of Q1-3SAT, Q2-3SAT, and Q3-3SAT in Section 2. Section 3 is devoted to the study of the computational complexity of the questions Q1, Q2, and Q3 for the set of vertex colorings of a graph.

2 Defining sets and SAT

Let D and R be two sets, and $f : D \rightarrow R$ be a function. We can refer to f as the set $f = \{(x, f(x)) : x \in D\}$. This representation enables us to study the defining set of a family of functions.

Let ϕ be a k CNF with variables $V = \{v_1, v_2, \dots, v_n\}$. For the sake of simplicity we use the notation $\phi(\mathbf{v})$ instead of $\phi(v_1, \dots, v_n)$, where we would think of \mathbf{v} as a vector of v_1, v_2, \dots, v_n . Since any truth assignment $t : V \rightarrow \{\text{true}, \text{false}\}$ is a function, we can study the defining set of a family of assignments. A **proper assignment** of ϕ is an assignment which makes ϕ true. Let $S \subseteq V$, be a subset of the variables of ϕ . A **partial assignment** of ϕ over the set S is a truth assignment $t : S \rightarrow \{\text{true}, \text{false}\}$. The set S is called the support set of t , and this is denoted by $S = \text{supp}(t)$. A partial assignment over S is called **proper** if every clause of ϕ contains at least one true literal from the variables in S .

For every k CNF ϕ , let $\mathcal{P}(\phi)$ denote the family of proper assignments of ϕ . We study the computational complexity of the general questions Q1, Q2, and Q3 for this special family. Let Q1- k SAT, Q2- k SAT, and Q3- k SAT stand for the three questions Q1, Q2, and Q3 in this case, respectively.

Duplicating a variable in a clause of a CNF does not change the family of its proper assignments. Hence if all clauses of a CNF are of size at most k (not necessarily equal to), then it can be converted to a k CNF. Therefore without loss of generality, we may always assume that all such expressions are in k CNF form.

In this section we show that Q1-3SAT is **CoNP**-Complete, and Q2-3SAT and Q3-3SAT are both Σ_2 -complete. From [14] we know that the following problem is Σ_2 -complete.

- $\exists \nexists$ 3SAT

INSTANCE: A 3CNF, $\phi(\mathbf{x}, \mathbf{y})$.

QUESTION: Is $\exists \mathbf{x} \nexists \mathbf{y} \phi(\mathbf{x}, \mathbf{y})$?

Next, we define the $\exists \exists!$ k SAT problem, and prove that it is Σ_2 -complete. A k CNF, ϕ consisting of variables $V = \{x_1, \dots, x_n, y_1, \dots, y_m\}$ with a proper partial assignment t over the set $\{y_1, y_2, \dots, y_m\}$ is given. The question is:

“Is there a partial assignment t' (not necessarily proper) over the set $\{x_1, \dots, x_n\}$ such that ϕ has a unique proper assignment r which satisfies $r(x_i) = t'(x_i)$ for every $1 \leq i \leq n$?”

Note that since t is a proper partial assignment, if such t' exists, then $r(y_j) = t(y_j)$ for $1 \leq j \leq m$.

- $\exists\exists!_* k\text{SAT}$

INSTANCE: A $k\text{CNF}$, $\phi(\mathbf{x}, \mathbf{y})$ and a proper partial assignment t over the set of the variables y_j .

QUESTION: Is $\exists\mathbf{x}\exists!_t \mathbf{y} \phi(\mathbf{x}, \mathbf{y})$?

Theorem 1 *The $\exists\exists!_* 4\text{SAT}$ problem is Σ_2 -complete.*

Proof. The problem is in Σ_2 . To prove the completeness, we give a reduction from $\exists\exists\exists 3\text{SAT}$. Consider a 3CNF , $\phi(\mathbf{x}, \mathbf{y})$ and the problem $\exists\mathbf{x}\exists\mathbf{y} \phi(\mathbf{x}, \mathbf{y})$. We construct an instance of $\exists\exists!_* 4\text{SAT}$, a 4CNF μ with a proper partial assignment t , as in the following. The expression μ has all of the variables of ϕ plus one more variable z . Let C_1, C_2, \dots, C_n be the clauses of $\phi(\mathbf{x}, \mathbf{y})$. Then

$$\mu(\mathbf{x}, \mathbf{y}, z) = (C_1 \vee z) \wedge (C_2 \vee z) \wedge \dots \wedge (C_n \vee z) \wedge (\bar{z} \vee y_1) \wedge (\bar{z} \vee y_2) \wedge \dots \wedge (\bar{z} \vee y_m) \quad (1)$$

The partial assignment $t(z) = \text{true}$ and $t(y_j) = \text{true}$ ($1 \leq j \leq m$) is given. This partial assignment is proper because there exists a true literal in every clause among z and y_j 's. For every proper assignment u of $\mu(\mathbf{x}, \mathbf{y}, z)$, if $u(z) = \text{true}$, then $u(y_j) = \text{true}$ ($1 \leq j \leq m$). If $u(z) = \text{false}$, then by ignoring the variable z in u , u is a proper assignment of $\phi(\mathbf{x}, \mathbf{y})$, and vice versa. So $\exists\mathbf{x}\exists\mathbf{y} \phi(\mathbf{x}, \mathbf{y})$ if and only if $\exists\mathbf{x}\exists!_t(\mathbf{y}, z) \mu(\mathbf{x}, \mathbf{y}, z)$. ■

Next, we modify the proof of Theorem 1 so that we can conclude that $\exists\exists!_* 3\text{SAT}$ is Σ_2 -complete. Consider $\mu(\mathbf{x}, \mathbf{y}, z)$, defined in (1). In $\mu(\mathbf{x}, \mathbf{y}, z)$ every clause of the form $(\bar{z} \vee y_j)$ has two literals. But a clause of the form $(C_i \vee z)$ has four literals. Suppose $C_i = a_1 \vee a_2 \vee a_3$, where a_1, a_2, a_3 are literals. We replace each clause $(C_i \vee z)$ in μ by C'_i defined as follows,

$$C'_i = (a_1 \vee a_2 \vee v_i) \wedge (a_3 \vee z \vee \bar{v}_i) \wedge$$

$$(\bar{a}_1 \vee \bar{z} \vee v_i) \wedge (\bar{a}_2 \vee \bar{z} \vee v_i) \wedge (\bar{a}_1 \vee \bar{a}_3 \vee v_i) \wedge (\bar{a}_2 \vee \bar{a}_3 \vee v_i),$$

where v_i 's are new variables, and call the new expression as $\mu'(\mathbf{x}, \mathbf{y}, \mathbf{v}, z)$. Thus

$$\mu'(\mathbf{x}, \mathbf{y}, \mathbf{v}, z) = C'_1 \wedge \dots \wedge C'_n \wedge (\bar{z} \vee y_1) \wedge (\bar{z} \vee y_2) \wedge \dots \wedge (\bar{z} \vee y_m).$$

Define the partial assignment as $u(z) = \text{true}$, $u(y_i) = \text{true}$, $u(v_i) = \text{true}$ for $1 \leq i \leq m$.

The following three observations imply that $\exists\mathbf{x}\exists!_t(\mathbf{y}, z) \mu(\mathbf{x}, \mathbf{y}, z)$ if and only if $\exists\mathbf{x}\exists!_u(\mathbf{y}, \mathbf{v}, z) \mu'(\mathbf{x}, \mathbf{y}, \mathbf{v}, z)$.

- (a) $u(z) = \text{true}$, $u(y_j) = \text{true}$ ($1 \leq j \leq m$), and $u(v_i) = \text{true}$ ($1 \leq i \leq n$) is a proper partial assignment of $\mu'(\mathbf{x}, \mathbf{y}, \mathbf{v}, z)$.
- (b) Every truth assignment to a_1, a_2, a_3 , and z which assigns a true value to at least one of them is extended uniquely to a proper assignment of C'_i .
- (c) Since every assignment which assigns a false value to a_1, a_2, a_3 , and z simultaneously is not a proper assignment of C'_i , every proper assignment of $\mu'(\mathbf{x}, \mathbf{y}, \mathbf{v}, z)$ leads to a proper assignment of $\mu(\mathbf{x}, \mathbf{y}, z)$ by ignoring the values of v_i 's.

Note that any proper subset of the clauses of C'_i does not satisfy these properties. For example consider the assignment $t(a_1) = \text{false}$, $t(a_2) = \text{true}$, $t(a_3) = \text{true}$, $t(z) = \text{false}$. In this case regardless of what value is assigned to v_i the first five clauses are satisfied, and the last clause is necessary to fix the value of v_i .

We conclude the following theorem from (a), (b), and (c).

Theorem 2 *The $\exists\exists!_*$ 3SAT problem is Σ_2 -complete.*

In the proof of Theorem 2 the problem $\exists\exists\exists$ 3SAT is reduced to $\exists\exists!_*$ 3SAT. In that proof by assuming that there are no variables x_i 's in $\phi(\mathbf{x}, \mathbf{y})$ (i.e. the number of variables after the first quantifier of $\exists\exists\exists$ 3SAT is zero), we can obtain a reduction from $\exists\exists\exists$ 3SAT to the problem which asks whether a given proper assignment of a 3CNF is its only proper assignment. This problem is a restriction of Q1-3SAT in which D , the set which is asked to be the defining set, is the empty set. Since $\exists\exists\exists$ 3SAT is **CoNP**-complete, we have:

Theorem 3 *Q1-3SAT is **CoNP**-complete.*

The next theorem determines the computational complexity of Q2-3SAT.

Theorem 4 *Q2-3SAT is Σ_2 -complete.*

Proof. The problem is in Σ_2 . We reduce $\exists\exists!_*$ 3SAT to this problem. Let $\exists\mathbf{x}\exists!_t\mathbf{y} \mu(\mathbf{x}, \mathbf{y})$ be an instance of $\exists\exists!_*$ 3SAT, where $\mu(\mathbf{x}, \mathbf{y})$ is a 3CNF with variables x_1, x_2, \dots, x_k and y_1, y_2, \dots, y_m , and t is a proper partial assignment over variables y_j . We construct an instance of Q2-3SAT, a 3CNF ϕ with a proper assignment t' , such that $(\mathcal{P}(\phi), t')$ has a defining set of size at most k , the number of the variables x_i , if and only if $\exists\mathbf{x}\exists!_t\mathbf{y} \mu(\mathbf{x}, \mathbf{y})$. In the following we describe how ϕ is obtained from $\mu(\mathbf{x}, \mathbf{y})$.

For every $1 \leq i \leq k$, consider two new variables v_i and v'_i , and replace every x_i in each clause of $\mu(\mathbf{x}, \mathbf{y})$ with v_i and every \bar{x}_i with v'_i .

For every $1 \leq j \leq m$, a literal a_j is defined as follows. If $t(y_j) = false$, then a_j is y_j and otherwise a_j is \bar{y}_j . We add the following clauses to the expression in which w_i are new variables.

$$(a_1 \vee a_2 \vee w_1) \wedge (\bar{w}_1 \vee a_3 \vee w_2) \wedge \dots \wedge (\bar{w}_{m-2} \vee a_m \vee w_{m-1}) \quad (2)$$

Note that by setting y_j 's according to the given assignment t , w_i 's are forced to take the truth value *true*. The following clauses are also added to the expression.

$$(\bar{w}_{m-1} \vee v_1 \vee \bar{v}'_1) \wedge (\bar{w}_{m-1} \vee \bar{v}_1 \vee v'_1) \wedge \dots \wedge (\bar{w}_{m-1} \vee v_k \vee \bar{v}'_k) \wedge (\bar{w}_{m-1} \vee \bar{v}_k \vee v'_k)$$

Call this new 3CNF, $\phi(\mathbf{v}, \mathbf{v}', \mathbf{y}, \mathbf{w})$. Let t' be the assignment $t'(v_i) = t'(v'_i) = false$ ($1 \leq i \leq k$), $t'(w_i) = true$ ($1 \leq i \leq m-1$), and $t'(y_j) = t(y_j)$ ($1 \leq j \leq m$). Note that t' is a proper assignment of ϕ .

We claim that $(\mathcal{P}(\phi), t')$ has a defining set of size at most k , if and only if $\exists \mathbf{x} \exists!_{t'} \mathbf{y} \mu(\mathbf{x}, \mathbf{y})$.

Suppose that $\exists \mathbf{x} \exists!_{t'} \mathbf{y} \mu(\mathbf{x}, \mathbf{y})$. This means that there is a partial assignment u over x_1, x_2, \dots, x_k such that the only proper values for y_j are the values that are assigned to them by the partial assignment t . If $u(x_i) = true$, we choose $(v'_i, false)$, and if $u(x_i) = false$, we choose $(v_i, false)$. Call this set S . We claim that S is a defining set of $(\mathcal{P}(\phi), t')$.

Suppose that S is not a defining set of $(\mathcal{P}(\phi), t')$. Then there is a proper assignment $t'' \neq t'$ which is an extension of S . Since $t''(v_i) = false$ and $t''(v'_i) = false$ for $v_i, v'_i \in \text{supp}(S)$, it can be easily seen that the assignment r defined as $r(x_i) = u(x_i)$ ($1 \leq i \leq k$) and $r(y_j) = t''(y_j)$ ($1 \leq j \leq m$) is a proper assignment to the variables of $\mu(\mathbf{x}, \mathbf{y})$ which is a contradiction. So all y_j take the values that are assigned to them by the assignment t . Hence w_i 's are true for all $1 \leq i \leq m-1$. Since the two clauses $(\bar{w}_{m-1} \vee v_i \vee \bar{v}'_i)$ and $(\bar{w}_{m-1} \vee \bar{v}_i \vee v'_i)$ are in ϕ , and exactly one of v_i or v'_i is in $\text{supp}(S)$, the value of the other one is also determined to be false, and this is the assignment t' .

Next suppose that $(\mathcal{P}(\phi), t')$ has a defining set S of size at most k . Then for every $1 \leq i \leq k$, at least one of v_i or v'_i is in $\text{supp}(S)$. Otherwise we can change the values of both v_i and v'_i to true, and still have a proper assignment. So a defining set of size at most k includes exactly one of $(v_i, false)$ or $(v'_i, false)$ for every $1 \leq i \leq k$. Let u be a partial assignment of $\mu(\mathbf{x}, \mathbf{y})$ such that $u(x_i) = true$ if $v_i \in \text{supp}(S)$, and $u(x_i) = false$ if $v'_i \in \text{supp}(S)$.

We claim that $\mu(\mathbf{x}, \mathbf{y})$ has a unique proper assignment r such that $r(x_i) = u(x_i)$ for every $1 \leq i \leq k$. Suppose that there is a proper assignment r for $\mu(\mathbf{x}, \mathbf{y})$ such that $r(x_i) = u(x_i)$ for all $1 \leq i \leq k$, but there exists at least one $1 \leq i_0 \leq m$ such that $r(y_{i_0}) \neq t(y_{i_0})$.

Consider $\phi(\mathbf{v}, \mathbf{v}', \mathbf{y}, \mathbf{w})$, and let $r'(y_j) = r(y_j)$ ($1 \leq j \leq k$). Since $r(y_{i_0}) \neq t(y_{i_0})$, it is possible to assign values $r'(w_i)$ ($1 \leq i \leq m-1$) such that $r'(w_{m-1}) = false$ and the clauses in (2) are true.

Note that *exactly* one of v_i or v'_i is in $\text{supp}(S)$. For every $1 \leq i \leq k$, if $v_i \in \text{supp}(S)$, then define $r'(v_i) = \text{false}$, $r'(v'_i) = \text{true}$; and if $v'_i \in \text{supp}(S)$, then define $r'(v_i) = \text{true}$, $r'(v'_i) = \text{false}$.

Since $t(w_{m-1}) = \text{false}$, the values assigned by r' do not make $(\bar{w}_{m-1} \vee v_i \vee \bar{v}'_i) \wedge (\bar{w}_{m-1} \vee \bar{v}_i \vee v'_i)$ false.

Now all clauses are satisfied. So there exists another proper assignment containing the defining set, which is a contradiction. ■

Theorem 5 *Q3-3SAT is Σ_2 -complete.*

Proof. The problem is in Σ_2 . We give a reduction from Q2-3SAT. Consider an instance of Q2-3SAT, a 3CNF ϕ with a proper assignment t and an integer k . Let the variables of ϕ be x_1, x_2, \dots, x_n . We add $n(k+1)$ new variables y_{ij} ($1 \leq i \leq n$ and $1 \leq j \leq k+1$). For every x_i , if $t(x_i) = \text{true}$, then we add the following clauses:

$$(\bar{x}_i \vee y_{i1}) \wedge (\bar{x}_i \vee y_{i2}) \wedge \dots \wedge (\bar{x}_i \vee y_{i(k+1)}),$$

and if $t(x_i) = \text{false}$, then we add the following clauses:

$$(x_i \vee y_{i1}) \wedge (x_i \vee y_{i2}) \wedge \dots \wedge (x_i \vee y_{i(k+1)}).$$

The new 3CNF consists of ϕ and these $n(k+1)$ new clauses. Denote this 3CNF by ϕ' . We claim that $\mathcal{P}(\phi')$ has a defining set of size at most k , if and only if $(\mathcal{P}(\phi), t)$ has a defining set of size at most k . Every defining set of $(\mathcal{P}(\phi), t)$ is also a defining set of $\mathcal{P}(\phi')$, because the assignment t forces all of the y_{ij} to take a true value.

Next suppose that there is a defining set of $\mathcal{P}(\phi')$ which fixes a proper assignment t' . For every $1 \leq x \leq n$, if $t'(x_i) \neq t(x_i)$, then since it is possible to assign every arbitrary values to $y_{i1}, y_{i2}, \dots, y_{i(k+1)}$, all these $k+1$ variables are in the defining set. Hence in every defining set of size at most k , all x_i take the same values in t' and t . Now, since $t'(x_i) = t(x_i)$, by fixing the value of x_i , the values of y_{ij} 's are determined to be true, for $1 \leq j \leq k+1$. So if $(y_{ij}, t'(y_{ij}))$ is in the defining set, then it is possible to replace it by $(x_i, t'(x_i))$. Thus a defining set of size at most k of $\mathcal{P}(\phi')$ can be modified so that all its elements are in $\{(x_i, t'(x_i)) : i = 1, \dots, n\}$, and $t'(x_i) = t(x_i)$. This is also a defining set of $(\mathcal{P}(\phi), t)$. ■

3 Vertex Coloring

For every graph G with vertex set $V = \{v_1, \dots, v_n\}$, every vertex coloring c of G is a function which maps every vertex v_i to a color $c(v_i)$. For every partial coloring c of G , define $\text{supp}(c)$ as the set of the vertices that c assigns a color to them. Denote the family of all $\chi(G)$ -vertex colorings of G by $\mathcal{C}(G)$. In [8] it is shown that the uniqueness of colorability is **CoNP**-complete. This implies the following theorem.

Theorem 6 *The problem Q1- VERTEX COLORING is CoNP-complete.*

In this section we show that both of the problems Q2 and Q3 for this family are Σ_2 -complete.

- Q2- VERTEX COLORING

INSTANCE: A graph G with a $\chi(G)$ -vertex coloring c , and an integer k .

QUESTION: Does $(\mathcal{C}(G), c)$ have a defining set of size at most k ?

- Q3- VERTEX COLORING

INSTANCE: A graph G , and an integer k .

QUESTION: Does $\mathcal{C}(G)$ have a defining set of size at most k ?

Theorem 7 *Q2- VERTEX COLORING is Σ_2 -complete for graphs with $\chi = 3$.*

Proof. The problem is in Σ_2 . To prove the completeness, we introduce a reduction from Q2-3SAT. Consider an instance of Q2-3SAT: A proper assignment t of $\phi(x_1, x_2, \dots, x_n)$ and an integer k . We construct a graph G_ϕ with chromatic number 3 and a 3-vertex coloring c_t of G_ϕ such that $(\mathcal{P}(\phi), t)$ has a defining set of size at most k if and only if $(\mathcal{C}(G_\phi), c_t)$ has a defining set of size at most $k + 4$.

We begin by considering a cycle of size 3 with vertices w_0, w_1 , and w_2 which are connected to four vertices w'_1, w'_2, w'_3 , and w'_4 as it is shown in Figure 1(a). For every variable x_i , add two vertices u_{x_i} and $u_{\bar{x}_i}$ and edges $\{u_{x_i}, u_{\bar{x}_i}\}$, $\{u_{x_i}, w_2\}$, and $\{u_{\bar{x}_i}, w_2\}$ to the graph. This is illustrated in Figure 1(a).

Consider a clause $C_i = (a_1 \vee a_2 \vee a_3)$ of ϕ , where a_j ($j = 1, 2, 3$) is a literal. Since t is a proper assignment of ϕ , without loss of generality we can assume that $t(a_2) = \text{true}$. For every such clause, we add a copy of the graph shown in Figure 1(b) to the graph, and connect its vertices to the other vertices as it is shown in Figure 1(b). Notice that u_{a_j} ($j = 1, 2, 3$) is one of the vertices $u_{x_1}, u_{x_2}, \dots, u_{x_n}$ or $u_{\bar{x}_1}, u_{\bar{x}_2}, \dots, u_{\bar{x}_n}$. Call this new graph as G_ϕ .

One can easily check that assigning a 3-coloring c_t to u_{a_1}, u_{a_2} , and u_{a_3} such that $c_t(w_0) = 0$, $c_t(w_1) = 1$, and $c_t(w_2) = 2$ and also $c_t(u_{a_2}) = 1$ determines the colors of $v_{i1}, v_{i2}, \dots, v_{i8}$ uniquely. Let c_t be a 3-coloring of G_ϕ defined as in the following:

- $c_t(w_0) = 0$, $c_t(w_1) = 1$, and $c_t(w_2) = 2$.
- $c_t(w'_1) = 1$, $c_t(w'_2) = 2$, $c_t(w'_3) = 0$, and $c_t(w'_4) = 2$.
- For every $1 \leq i \leq n$ if $t(x_i) = \text{true}$, then $c_t(u_{x_i}) = 1$ and $c_t(u_{\bar{x}_i}) = 0$, and otherwise $c_t(u_{x_i}) = 0$ and $c_t(u_{\bar{x}_i}) = 1$.
- Colors of v_{ij} are determined uniquely by the colors of the vertices above.

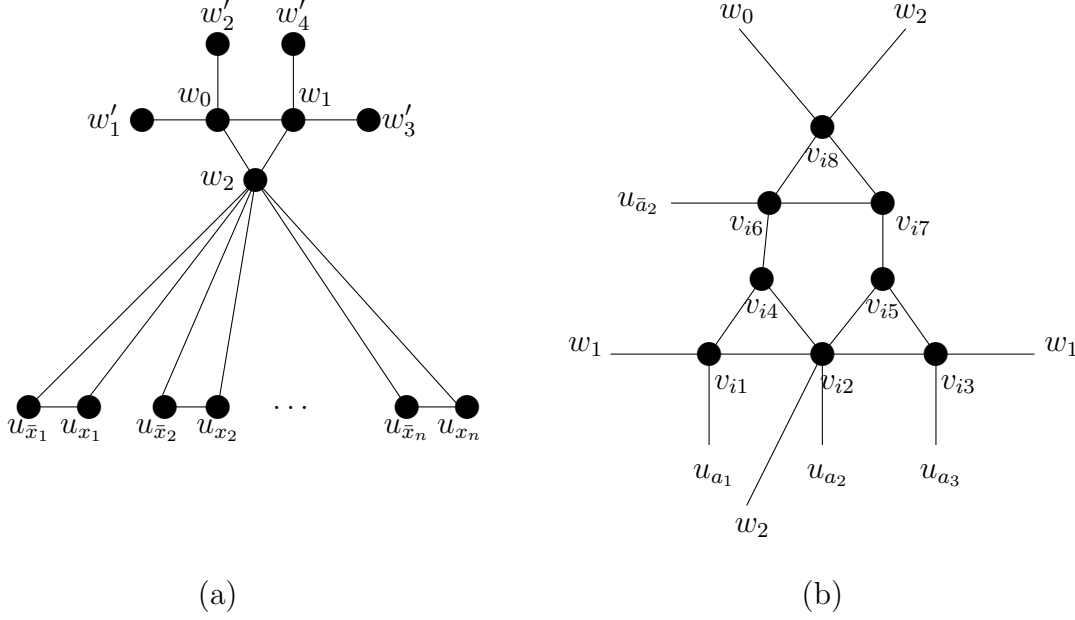


Figure 1: (a) The vertices u_{x_i} and $u_{\bar{x}_i}$ are connected to w_2 . (b) For every clause we add a copy of this graph to G_ϕ .

The vertices w'_1, w'_2, w'_3 , and w'_4 are in every defining set (otherwise we can change their colors). The colors of these four vertices determine the colors of w_0, w_1 , and w_2 uniquely.

We claim that the size of the smallest defining set of $(\mathcal{C}(G_\phi), c_t)$ is equal to the size of the smallest defining set of $(\mathcal{P}(\phi), t)$ plus 4. Note that any partial coloring which only assigns 0 or 1 to $u_{a_1}, u_{a_2}, u_{a_3}$ and does not assign 0 to all of them can be extended to a proper coloring of the graph in Figure 1(b). Moreover if all the vertices $u_{a_1}, u_{a_2}, u_{a_3}$ are colored by 0, then it can be easily seen that v_{i8} is also forced to be colored by 0. Since v_{i8} is connected to w_0 and w_2 , G_ϕ admits a 3-coloring, if and only if ϕ has a proper assignment.

Suppose $(\mathcal{C}(\phi), t)$ has a defining set consists of k variables $x_{i_1}, x_{i_2}, \dots, x_{i_k}$. Then assigning colors of $k + 4$ vertices w'_1, w'_2, w'_3, w'_4 and $u_{x_{i_1}}, u_{x_{i_2}}, \dots, u_{x_{i_k}}$ constitutes a defining set of $(\mathcal{C}(G_\phi), c_t)$.

Next suppose that S is the smallest defining set of $(\mathcal{C}(G_\phi), c_t)$. Then w'_1, w'_2, w'_3, w'_4 are in $\text{supp}(S)$. By assigning the colors of these vertices the colors of w_1, w_2, w_3 , and all v_{i8} 's are determined uniquely. It can be verified easily that for every clause $C_i = (a_1, a_2, a_3)$ of ϕ , since $c_t(u_{a_2}) = 1$, the colors of $v_{i1}, v_{i2}, \dots, v_{i7}$ are determined uniquely by fixing the color of u_{a_2} , and the color of u_{x_i} determines the color of $u_{\bar{x}_i}$. Hence we can assume that $\text{supp}(S)$ contains w'_1, w'_2, w'_3, w'_4 , and some of u_{x_i} . Using

the fact that any partial coloring which only assigns 0 or 1 to $u_{a_1}, u_{a_2}, u_{a_3}$ and does not assign 0 to all of them can be extended to a proper coloring of the graph in Figure 1(b), we conclude that the corresponding variables of these u_{x_i} constitute a defining set of $(\mathcal{C}(\phi), t)$. ■

Theorem 8 *Q3- VERTEX COLORING is Σ_2 -complete for graphs with $\chi = 3$.*

Proof. The problem is in Σ_2 . We give a reduction from Q2- VERTEX COLORING when $\chi = 3$. Consider an instance $(\mathcal{C}(G), c)$ of Q2- VERTEX COLORING, where G is a graph and c is a 3-vertex coloring of G . Assume that the range of c is the set $\{0, 1, 2\}$. An integer k is given, and it is asked that “Is there a defining set of size at most k for $(\mathcal{C}(G), c)$?” We construct a new graph H as follows:

1. First let H be the disjoint union of G and a cycle $w_0w_1w_2$ of size 3. Then
2. for every vertex u_i of G , let c_1 and c_2 be the two colors other than $c(u_i)$. Add $2k + 2$ vertices $v_{u_i, c_j, 1}, v_{u_i, c_j, 2}, \dots, v_{u_i, c_j, k+1}$ ($1 \leq j \leq 2$) to H . For every $1 \leq t \leq k + 1$, connect $v_{u_i, c_j, t}$ to both u_i and w_{c_j} . (Notice that w_{c_j} is one of w_0, w_1 , or w_2 .)
3. Add four new vertices w'_1, w'_2, w'_3 , and w'_4 to H , and connect w'_1 and w'_2 to w_0 , and also w'_3 and w'_4 to w_1 .

Now we claim that $\mathcal{C}(H)$ has a defining set of size at most $k + 4$ if and only if $(\mathcal{C}(G), c)$ has a defining set of size at most k .

First consider a defining set of size at most k for $(\mathcal{C}(G), c)$, say D . If we fix the colors of the vertices in D and assign the colors 1 to w'_1 , 2 to w'_2 , 0 to w'_3 , and 2 to w'_4 , then these $k + 4$ vertices constitute a defining set of $\mathcal{C}(H)$.

Next suppose that D is the smallest defining set of $\mathcal{C}(H)$ which has at most $k + 4$ vertices. Without loss of generality assume that in the extension of D to a 3-vertex coloring c' of H , w_i ($0 \leq i \leq 2$) is colored by i . Since the degrees of w'_1, w'_2, w'_3 , and w'_4 are equal to one, they are in $\text{supp}(D)$. Suppose that in the extension of D to a 3-coloring of H , a vertex u_i of G is colored by $c'(u_i)$ which is not equal to $c(u_i)$. The vertices $v_{u_i, c'(u_i), t}$ ($1 \leq t \leq k + 1$) are only connected to u_i and $w_{c'(u_i)}$. Since these two vertices are colored by the same colors, all these $k + 1$ vertices are in the defining set, and with the four vertices w'_i , the size of the defining set is at least $k + 5$. Since D is of size at most $k + 4$, for every vertex u_i , $c'(u_i) = c(u_i)$.

We can suppose that w'_1 and w'_2 (and so w'_3 and w'_4) are colored by different colors. Otherwise by changing the color of w'_2 , (and so w'_4), D still remains a defining set of $\mathcal{C}(H)$. Since w'_1 and w'_2 and also w'_3 and w'_4 are colored by different colors, they determine the colors of w_1, w_2 , and w_3 uniquely. Therefore since D is the smallest defining set, none of w_1, w_2 , and w_3 is in $\text{supp}(D)$. Also if a vertex $v_{u_i, c_j, t}$ ($c_j \neq c(u_i)$) is in D , then we can replace it with u_i . Now, if we remove the four vertices w'_i from the defining set, we obtain a defining set of $(\mathcal{C}(G), c)$. ■

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